

Some properties of quantum reliability function for quantum communication channel

K. Kurokawa, *M. Sasaki, M. Osaki, and O. Hirota

*Research Center for Quantum Communications,
Tamagawa University, Tokyo 194 (Japan)*

** Communication Research Laboratory,
Ministry of Posts and Telecommunications*

Abstract

This paper presents some examples of quantum reliability function for the quantum communication system in which classical information is transmitted by quantum states. In addition, the quantum Cut off rate is defined. They will be compared with Gallager's reliability function for the same system.

1 Introduction

Recently, quantum communication and information theory has received much attention, because many distinguished features as quantum nature were discovered in comparison with conventional information theory. It is well known that Shannon's first and second theorems play the most important role in the conventional one, which give an existence of effective coding schemes in information source and channel[1]. Recently, the quantum version of Shannon's first theorem was proved by Schumacher[2]. On the second theorem, Stratonovich[3] and Holevo[4] pointed out that a true capacity for quantum measurement channel is greater than the maximum mutual information with respect to detection operators and a priori probability for information symbols, and that it will be bounded by von Neumann entropy for an ensemble of signal states. Hausladen et. al[5] proved quantum version of Shannon's second theorem by introducing a typical sub-space in addition to random coding scheme and square root measurement. This corresponds to a typical sequence in the conventional theory. As a result, it was shown that the zero error channel capacity is von Neumann entropy for an ensemble of signal states. On the other hand, we remain problems how to derive the maximum mutual information with respect to detection operators and a priori probability for information symbols so called un-coded capacity : C_1 . In this problem, Holevo's necessary condition formula[6] and Davies's theorem[7] play an important role. Fuchs et. al [8] tried to give some examples of C_1 , and Ban[9] and Osaki[10] proved that if the signal states are group covariant, then the square root measurement and minimax solution in the detection theory satisfy Holevo's necessary condition, and showed some examples of C_1 . Also Sasaki[11] recently gave some example for super additivity of capacity and its coding scheme in order to connect a gap between C_1 and von Neumann entropy as the true channel capacity. As a natural extension, Holevo[12] presented a theory of quantum reliability function corresponding to Gallager's reliability function in the conventional theory[13]. The theory of reliability function is

very important in communication engineering. So it is also essential to clarify detailed properties of quantum reliability function in order to study coding scheme with finite length. In this paper, we shall show numerical examples of Holevo's quantum reliability function for several quantum state signals, and compare with Gallager's reliability function to quantum measurement channel.

2 A theory of quantum reliability function

2.1 Reliability function

Here we survey a theory of reliability function for quantum channel. Let us consider quantum channel with a finite input pure state: $\rho_i = |\psi_i\rangle\langle\psi_i|$. According to quantum version of Shannon's second theorem by Hausladen, the zero error channel capacity is given by von Neumann entropy as follows:

$$H(S_\xi) = -\text{Tr}[S_\xi \ln S_\xi] \quad (1)$$

$$S_\xi = \sum \xi_i \rho_i \quad (2)$$

The von Neumann entropy for quantum information source is larger than C_1 . In order to take into account this fact in reliability function theory, the next formalism can be used.

$$E_Q(R) = \lim_{n \rightarrow \infty} \sup \frac{1}{n} \ln \frac{1}{P_e} \quad (3)$$

where P_e is average error probability of code words, and R is information transmission rate. One cannot calculate $E_Q(R)$ directly. However, if there is a bound for this $E_Q(R)$, then one can estimate the following inequality.

$$P_e \leq e^{-nE_Q(R)} \leq e^{-nE_{Qr}(R)} \quad (4)$$

An upper bound for the average error probability for code words was derived based on square root measurement and random coding technique by Holevo[12].

<Theorem>

For any M and $0 \leq s \leq 1$, the following upper bound is valid:

$$P_e \leq 2(M-1)^s [\text{Tr} S_\xi^{1+s}]^n \quad (5)$$

In the above theorem, we can set

$$M = e^{nR} > 1 \text{ Fcode word.}$$

$$S_\xi \text{ Fdensity operator.}$$

$$\left[\text{Tr} S_\xi^{1+s}\right]^n = e^{n \ln \text{Tr} S_\xi^{1+s}} \quad (6)$$

When one inserts the above relations into Eq(5), we have

$$P_e \leq 2e^{nsR} \times e^{n \ln \text{Tr} S_\xi^{1+s}} = 2e^{n\{\ln \text{Tr} S_\xi^{1+s} + sR\}} \quad (7)$$

Let us rewrite the above equation as follows:

$$P_e \leq 2e^{-n\{\mu(s, \xi) - sR\}} \quad (8)$$

where

$$\mu(s, \xi) = -\ln \text{Tr} S_\xi^{1+s} \quad (9)$$

As a result, we can define

$$E_{Qr}(R) \equiv \max_s \max_\xi [\mu(s, \xi) - sR] \quad (10)$$

The maximization with respect to "s" is given by

$$\frac{\partial}{\partial s} [\mu(s, \xi) - sR] = \frac{\partial [\mu(s, \xi)]}{\partial s} - R = 0 \quad (11)$$

where we have

$$\frac{\partial \mu(s, \xi)}{\partial s} = \frac{-\text{Tr} S_\xi^{1+s} \ln S_\xi}{\text{Tr} S_\xi^{1+s}} = \frac{-\sum \lambda_j^{1+s} \ln \lambda_j}{\sum \lambda_j^{1+s}} \quad (12)$$

For the maximization with respect to a priori probability, it is natural way.

2.2 Cut off rate

In the conventional theory, we define sometimes Cut off rate which is the special case of reliability function [14]. Here we can define also quantum version of Cut off rate based on Eq(10) as follows:

$$R_0 \equiv \max_\xi \mu(\xi, s=1) = \max -\ln \text{Tr} S_\xi^2 = \max -\ln \sum \sum \xi_i \xi_j |\langle \psi_i | \psi_j \rangle|^2 \quad (13)$$

This is exactly a maximization of the entropy introduced by Stratonovich [3]. The maximization is very simple. That is, one needs only to optimize a priori probability as same with classical one. In the quantum reliability function, let us assume that the signal power is enough large, which corresponds to almost orthogonal states. In this case, the optimization with respect to "s" does not make sense. So the quantum reliability function in the case with large photon signals may be replaced by the quantum Cut off rate.

In future problems, when we encounter more difficult problems in which calculation of reliability function like in the conventional theory is so difficult, the quantum cut off rate will provide useful way.

3 Examples for several quantum state signals.

Let us here calculate quantum version of reliability function for several quantum signals. For our purpose, the next theorem is useful.

<Theorem>[3]

The eigenvalues of density operator for ensemble of signal states are equal to those of the Gram matrix for signal set.

(A) Binary, 3 ary PSK and 4 ary orthogonal signal by pure state

Let us first assume that the signal states are $|\alpha\rangle, |-\alpha\rangle$. The Gram matrix for this signal is given by

$$\begin{aligned} & \begin{bmatrix} \sqrt{\xi_1} \langle \alpha | \alpha \rangle \sqrt{\xi_1} & \sqrt{\xi_1} \langle \alpha | -\alpha \rangle \sqrt{\xi_2} \\ \sqrt{\xi_2} \langle -\alpha | \alpha \rangle \sqrt{\xi_1} & \sqrt{\xi_2} \langle -\alpha | -\alpha \rangle \sqrt{\xi_2} \end{bmatrix} \\ &= \begin{bmatrix} \xi_1 & \kappa \sqrt{\xi_1 \xi_2} \\ \kappa \sqrt{\xi_1 \xi_2} & \xi_2 \end{bmatrix} \end{aligned} \quad (14)$$

where

$$\kappa = \langle \alpha | -\alpha \rangle = \exp \left[-2|\alpha|^2 \right].$$

The eigenvalues of the above matrix are

$$\lambda_1 = \frac{1}{2} \left[1 - \sqrt{1 - 4(1 - \kappa^2)\xi(1 - \xi)} \right] \quad (15)$$

$$\lambda_2 = \frac{1}{2} \left[1 + \sqrt{1 - 4(1 - \kappa^2)\xi(1 - \xi)} \right] \quad (16)$$

As a result, we have the next relation.

$$\mu_r(\xi) \equiv -\ln \left\{ \lambda_1^{1+s} + \lambda_2^{1+s} \right\} - sR = \mu(s, \xi) - sR \quad (17)$$

$$\frac{\partial \mu(s, \xi)}{\partial s} = \frac{-\lambda_1^{1+s} \ln \lambda_1 - \lambda_2^{1+s} \ln \lambda_2}{\lambda_1^{1+s} + \lambda_2^{1+s}} - R = 0 \quad (18)$$

From Eqs (16) and (17), we can find quantum reliability function for this signals. The figure 1-a shows a numerical example. Then, based on the same procedure mentioned above, we can give the quantum reliability functions for 3 ary PSK and M ary orthogonal signal systems. Figure 1-b and 1-c show the numerical examples for them. The signal states of 3-aryPSK and orthogonal signal are as follows:

$$|\alpha\rangle, \left| \alpha e^{i\frac{2}{3}\pi} \right\rangle, \left| \alpha e^{-i\frac{2}{3}\pi} \right\rangle \quad (19)$$

$$\begin{aligned}
|\psi_1\rangle &= |\alpha\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4 \\
|\psi_2\rangle &= |0\rangle_1 |\alpha\rangle_2 |0\rangle_3 |0\rangle_4 \\
|\psi_3\rangle &= |0\rangle_1 |0\rangle_2 |\alpha\rangle_3 |0\rangle_4 \\
|\psi_4\rangle &= |0\rangle_1 |0\rangle_2 |0\rangle_3 |\alpha\rangle_4
\end{aligned} \tag{20}$$

In these figures, solid line means the part of quantum reliability function with optimum value $s = 1$. Dashed line means that with optimum value $s < 1$.

(B) Ternary pure states signal

Here let us consider ternary pure states given as $|0\rangle, |\alpha\rangle, |-\alpha\rangle$. Ternary signal is non-symmetric and the optimization with respect to a priori probability is necessary[15]. Hence, in this case, we should clarify the maximum von Neumann entropy, because the maximum value is not given by equal a priori probability, while equal a priori probability gives the maximum von Neumann entropy in cases of binary, PSK, and orthogonal signal systems. The Gram matrix for this signal is given by

$$\begin{bmatrix} \xi_1 & \kappa\sqrt{\xi_1\xi_2} & \kappa\sqrt{\xi_1\xi_3} \\ \kappa\sqrt{\xi_2\xi_1} & \xi_2 & \kappa^4\sqrt{\xi_2\xi_3} \\ \kappa\sqrt{\xi_3\xi_1} & \kappa^4\sqrt{\xi_3\xi_2} & \xi_3 \end{bmatrix} \tag{21}$$

$$\kappa = e^{-\frac{N_s}{2}}$$

$$N_s = |\alpha|^2 \text{ Fsignal photon.}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of the Gram matrix. Then the von Neumann entropy becomes

$$H(S_\xi) = -\lambda_1 \ln \lambda_1 - \lambda_2 \ln \lambda_2 - \lambda_3 \ln \lambda_3 \tag{22}$$

The von Neumann entropy as function of a priori probabilities can be visualized as shown in figures 2-a and 2-b. Thus, a priori probabilities which give the maximum von Neumann entropy are not uniform. By taking into account this fact, we show the quantum reliability function for ternary signals in figure 3.

4 Application of Gallager's reliability function.

Here we apply Gallager's reliability function to quantum channel. In general it is defined as follows:

$$E(R) = \max_{\rho, \xi} E' \tag{23}$$

$$E' = -\rho R - \ln \sum_{j=1}^k \left(\sum_{i=1}^r \xi_i p_{ij}^{1/(1+\rho)} \right)^{1+\rho} \tag{24}$$

$$\begin{aligned}
0 &< \rho \leq 1, \quad 0 \leq \xi_i \leq 1, \\
\xi &= \{\xi_1, \dots, \xi_r\} \text{ Fa priori probability,} \\
k &\text{Fnumber of output symbol,} \\
r &\text{Fnumber of input symbol.}
\end{aligned}$$

In order to apply Gallager's reliability function to quantum system, we first have to define channel matrix . As an example, we here treat the binary case. In this case, we employ the optimum detection operators for average error probability. So the channel matrix can be represented as follows:

$$\begin{bmatrix} \frac{2F+\lambda-2\kappa^2+1}{4F} & \frac{2F-\lambda+2\kappa^2-1}{4F} \\ \frac{2F-\lambda+2\lambda\kappa^2-1}{4F} & \frac{2F+\lambda-2\lambda\kappa^2+1}{4F} \end{bmatrix} \quad (25)$$

where

$$F = \sqrt{\left(\frac{1+\lambda}{2}\right)^2 - \lambda\kappa^2}, \quad \lambda = \xi_1/\xi_2, \\ \kappa = \langle \alpha | -\alpha \rangle = \exp[-2|\alpha|^2].$$

From Eqs(23), (24), we can find Gallager's reliability function. In figure 4, we show the numerical property in comparison with the quantum case. As a result, we obtaine that the zero error capacity in case of Gallager is C_1 , and it cannot achieve von Neumann entropy.

5 Conclusions

In this paper, we surveyed a theory of quantum reliability function and defined quantum Cut off rate. Then we examined quantum reliability function for several quantum signals. As a result, if signal power is enough large, the quantum reliability function is almost same as quantum cut off rate. In addition , it was shown that Gallager's functions for quantum systems provide only a coding scheme based on un-coded capacity C_1 .

References

- [1] T. M. Cover and J. A. Thomas, "Information Theory," John Wiley & Sons, Inc, 1991.
- [2] M. Schumacher, Phys. Rev. **A51**, 2738, 1995.
- [3] R. L. Stratonovich and A. G. Vantsyan, Probl. Upr. Teor. Inf. , **7**, No.3, 161, 1978.
- [4] A. S. Holevo, Probl. Per. Inf. ,**15**, No4, 247, 1979.
- [5] P. Housladen, R. Jozsa, B. Schumacher, M. Westmoreland, and W. K. Wootters, Phys. Rev. **A54**, 1869, 1996.
- [6] A. S. Holevo, Probl. Inf. ,**9**, 177, 1973.
- [7] E. Davies, IEEE. Trans. **IT-24**, 596, 1978.
- [8] C. Fuchs, and A. Peres, Phys. Rev. **A53**, 2038, 1996.

- [9] M. Ban, K. Kurokawa, R. Momose, and O. Hirota, International, J. of Theoretical Phys. (to be published), 1997(July).
- [10] M. Osaki, M. Ban, and O. Hirota, J. of Mod. Opt. (to be published), 1997.
- [11] M. Sasaki, K. Kato, M. Izutsu, and O. Hirota, quant-ph/**9705043**, 1997.
- [12] M. V. Burnashev and A. S. Holevo, quant-ph/**973013v2**, 1997.
- [13] R. G. Gallager, "Information Theory and Reliable Communication,". Jon Wiley & Sons, Inc, 1968.
- [14] J.Massey, IEEE, Trans. **COM-29**, 1615, 1981.
- [15] M.Osaki, M.Ban, and O.Hirota, Phys. Rev. **A54**, 1691, 1996.

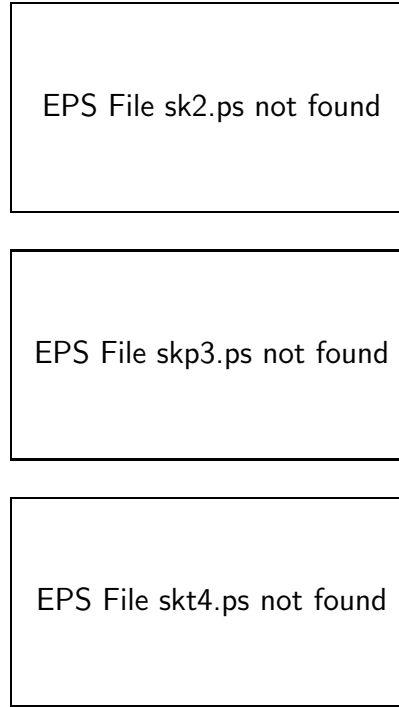


Fig.1 Numerical properties of quantum reliability function for binary(1-a), 3-ary PSK(1-b), and 4-ary orthogonal signal(1-c).

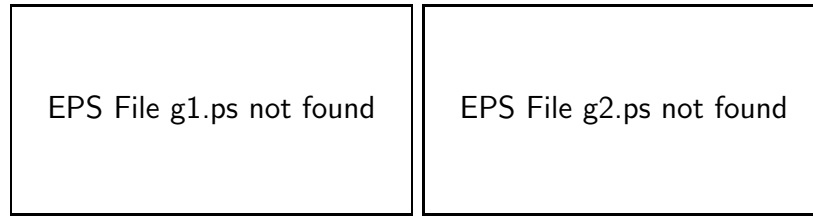


Fig.2 Numerical property of von Neumann entropy.

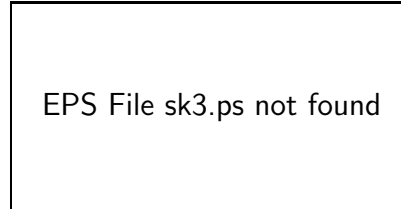


Fig.3 Quantum reliability function for ternary.

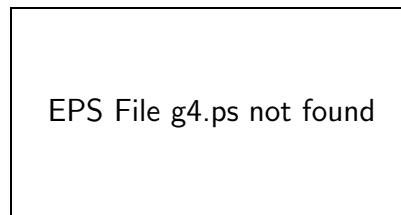


Fig.4 Comparison with quantum and Gallager reliability functions.

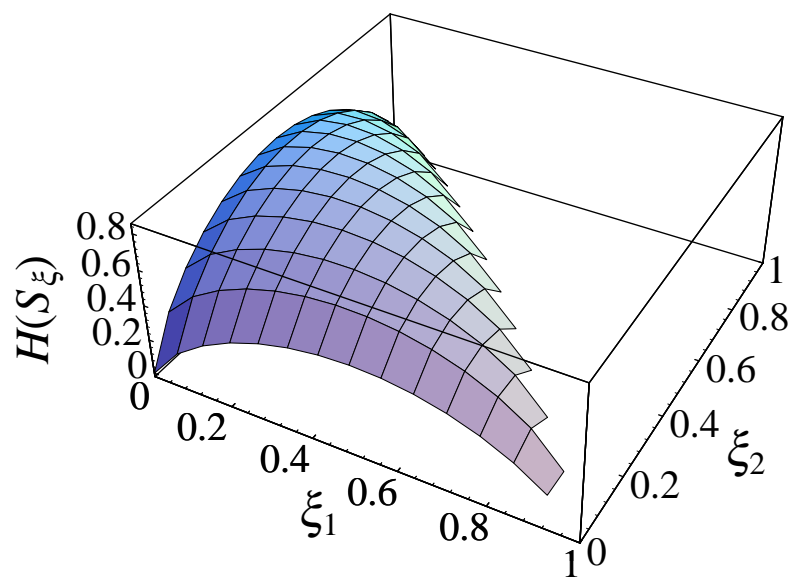


Fig. 2 – a

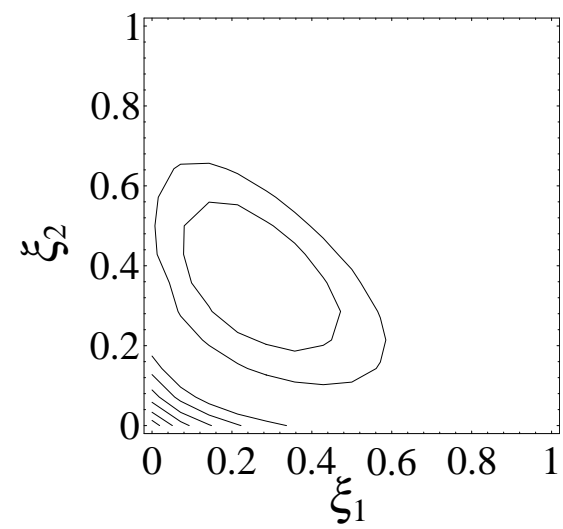


Fig. 2 – b

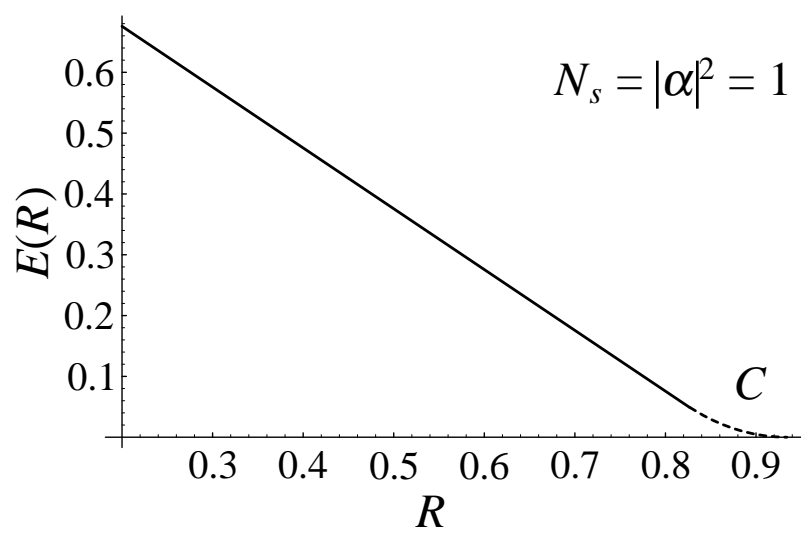


Fig. 1 – a

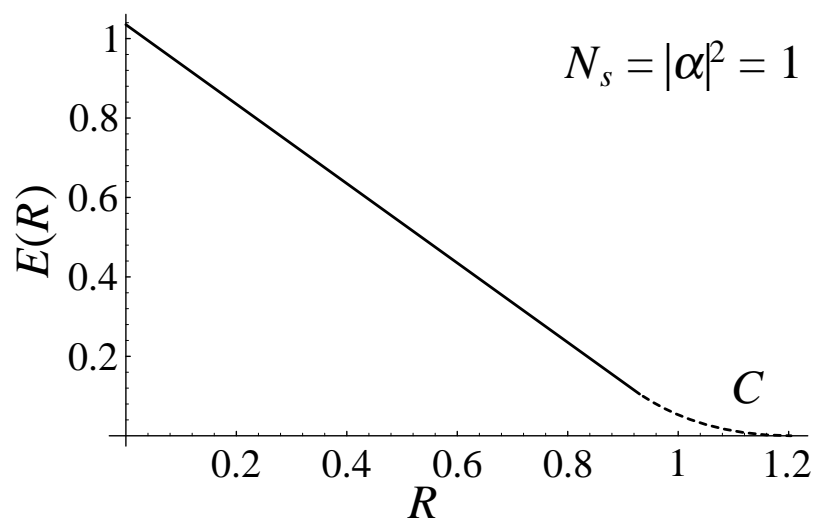


Fig. 3

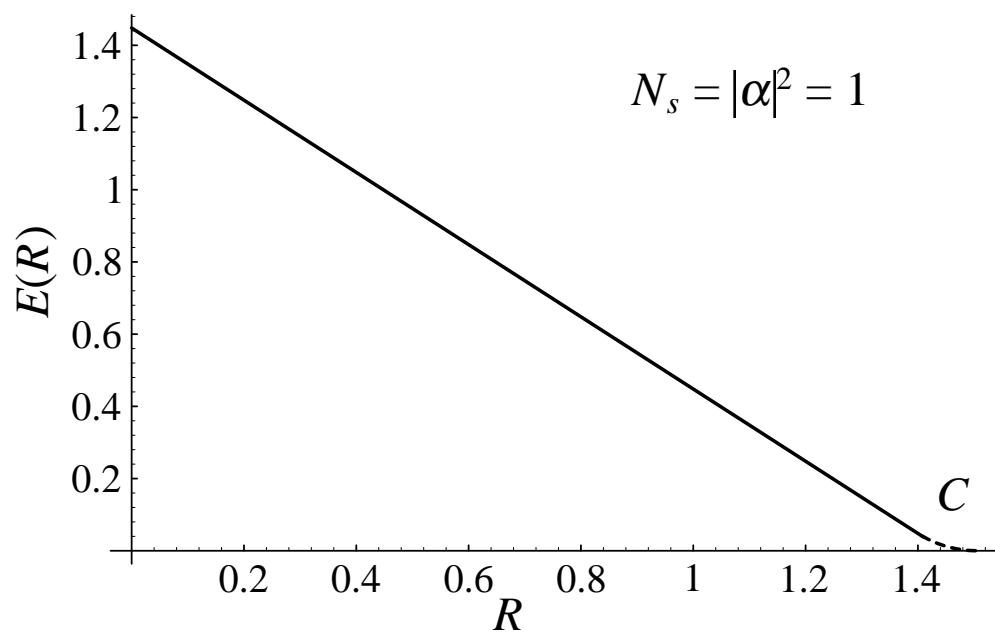


Fig. 1 – b

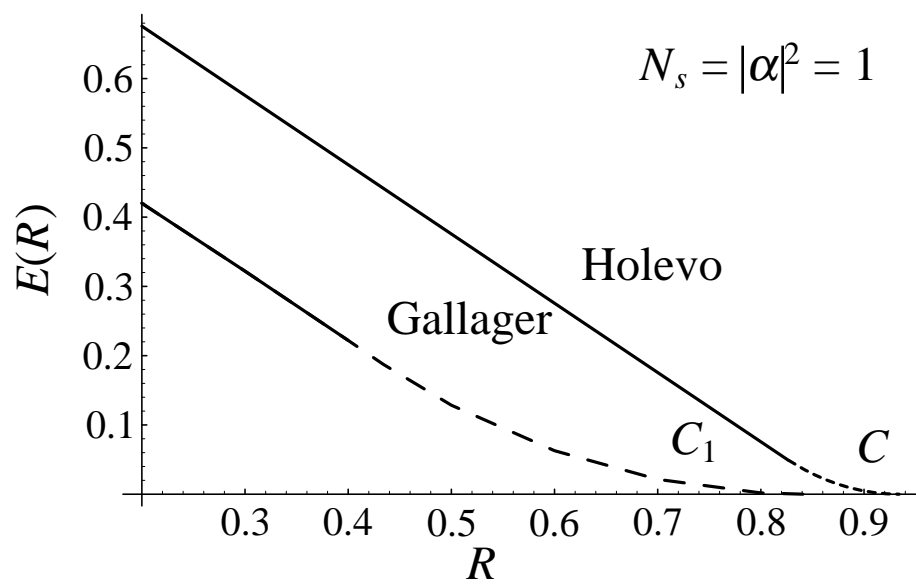


Fig. 4

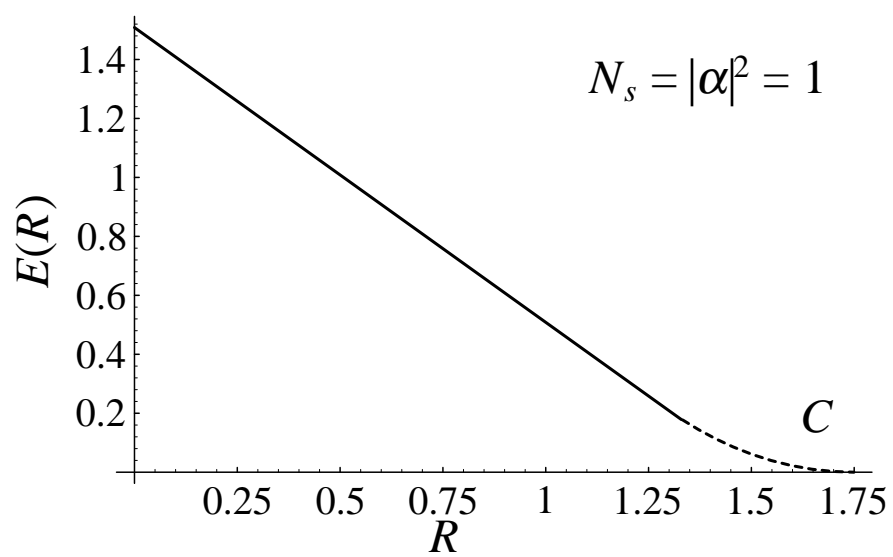


Fig. 1 – c